# Remarks on the Structure of Dirichlet Forms on Standard Forms of von Neumann Algebras

Yong Moon Park

Department of Mathematics and Institute for Mathematical Sciences,
Yonsei University, Seoul 120-749, Korea
E-mail : ympark@yonsei.ac.kr

**Abstract**

For a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\xi_0$, we investigate the structure of Dirichlet forms on the natural standard form associated with the pair $(\mathcal{M}, \xi_0)$. For a general Lindblad type generator $L$ of a conservative quantum dynamical semigroup on $\mathcal{M}$, we give sufficient conditions so that the operator $H$ induced by $L$ via the symmetric embedding of $\mathcal{M}$ into $\mathcal{H}$ to be self-adjoint. It turns out that the self-adjoint operator $H$ can be written in the form of a Dirichlet operator associated to a Dirichlet form given in [23]. In order to make the connection possible, we also extend the range of applications of the formula in [23].

## 1 Introduction

The purpose of this work is to investigate the structure of Dirichlet forms on a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\xi_0$. We are looking for a connection between Lindblad type generators of conservative quantum dynamical semigroup(q.d.s.) on $\mathcal{M}$ [19] and Dirichlet operators associated to Dirichlet forms introduced in [23]. In order to make the connection possible we first extend the range of applications of the formula of Dirichlet forms in [23]. We then consider a general Lindblad type generator $L$ of a conservative q.d.s. on $\mathcal{M}$. We give sufficient conditions under which the operator $H$ induced by $L$ via the symmetric embedding of $\mathcal{M}$ into $\mathcal{H}$ is self-adjoint. It turns out that the self-adjoint operator $H$ can be expressed in the form of a Dirichlet operator associated to a Dirichlet form given in [23]. In this sense, the Dirichlet forms constructed in [23] can be considered to be natural.

The need to construct Markovian semigroups on von Neumann algebras, which are symmetric with respect to a non-tracial state, is clear for various applications to open systems[14], quantum statistical mechanics[10] and quantum probability[4, 1,

24], Although on the abstract level we have quite well-developed theory[13, 16, 17], the progress in concrete applications is very slow. One of the reasons is that the general structure of Dirichlet forms for non-tracial states is not well-understood compared to the tracial case[2, 3, 6, 12]. For constructions of Dirichlet forms for non-tracial states, we refer to [8, 9, 11, 18, 20, 21, 25, 23] and the references there in. In [23], we gave a general construction method of Dirichlet forms on standard forms of von Neumann algebras. The method has been used to construct (translation invariant) symmetric Markovian semigroups for quantum spin systems[23], the CCR and CAR algebras with respect to quasi-free states[8, 9] and quantum mechanical systems[7].

Let us describe the content of this paper briefly. Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\xi_0$ for $\mathcal{M}$. Let $\Delta$ and $J$ be the modular operator and modular conjugation respectively associated with the pair $(\mathcal{M}, \xi_0)$[10]. Denote by $\sigma_t$, $t \in \mathbb{R}$, the group of modular automorphisms : $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$, $A \in \mathcal{M}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $*$-isomorphisms defined by $j(A) := JAJ$, $A \in \mathcal{M}$, where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$. For any $\lambda > 0$, denote by $\mathcal{M}_\lambda$ the dense subset of $\mathcal{M}$ consisting every $\alpha_t$-analytic element of $\mathcal{M}$ with a domain containing the strip $\{z \in \mathbb{C} : |Im\, z| \leq \lambda\}$.

As mentioned before, we are looking for a connection between Lindblad type generators and Dirichlet operators associated to Dirichlet forms constructed in [23]. To make a connection possible, we need to extend the range of applications of the formula given in [23]. In [23], we constructed a Dirichlet form for any $x \in \mathcal{M}_{1/4}$ and admissible function $f$ [23, Theorem 3.1]. In this paper, we consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ given by

$$f_0(t) = 2(e^{2\pi t} + e^{-2\pi t})^{-1}. \tag{1.1}$$

The function $f_0$ will play a special role. We extend the construction of Dirichlet forms to the function $f_0$ in Theorem 2.1.

We next consider the generators of conservative q.d.s. on $\mathcal{M}$. The most natural generator would be the following Lindblad type generator [19, 24] :

$$L(A) = \sum_{k=1}^{\infty} \{y_k^* y_k A - 2y_k^* A y_k + A y_k^* y_k\} + i[Q, A], \quad A \in \mathcal{M}, \tag{1.2}$$

where $y_k \in \mathcal{M}$, $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} y_k^* y_k$ converges strongly. Here we have used the notation $[A, B] := AB - BA$, $\forall A, B \in \mathcal{M}$. However, in order to avoid the convergence problems in the study of (1.2)(see Remark 2.2 (c)), we concentrate to the case in which only finite $y_k$'s in (1.2) are not zero.

For given $\{y_1, y_2, \cdots, y_n\} \subset \mathcal{M}_{1/2}$ and $Q = Q^* \in \mathcal{M}_{1/2}$, we consider the following Lindblad type generator $L$ of a conservative q.d.s. :

$$\begin{aligned} L &: \mathcal{M} \to \mathcal{M}, \\ L(A) &= \sum_{k=1}^{n} y_k^* y_k A - 2y_k^* A y_k + A y_k^* y_k + i[Q, A], \quad A \in \mathcal{M}. \end{aligned} \tag{1.3}$$

Consider the following symmetric embedding [13]:

$$\begin{aligned} i_0 &: \mathcal{M} \to \mathcal{H}, \\ i_0(A) &= \Delta^{1/4} A \xi_0, \ A \in \mathcal{M}, \end{aligned}$$

and define the operator $H$ on $\mathcal{H}$ by

$$H\Delta^{1/4} A\xi_0 = \Delta^{1/4} L(A)\xi_0, \quad A \in \mathcal{M}. \tag{1.4}$$

If $H$ is self-adjoint, $H$ generates a symmetric Markovian semigroup on $\mathcal{H}$[13].

Let $L : \mathcal{M} \to \mathcal{M}$ be given as (1.3). Put $x_k := \sigma_{i/4}(y_k), \ k = 1, 2, \cdots, n$. Assume that the following property holds:

$$\sum_{k=1}^{n} x_k j(x_k) = \sum_{k=1}^{n} x_k^* j(x_k^*). \tag{1.5}$$

Then the operator $H$ associated to $L$ by the relation (1.4) is self-adjoint if and only if $Q$ is given by

$$Q = \sum_{k=1}^{n} Q_k \tag{1.6}$$

where

$$Q_k = i \int \sigma_t \left( x_k^* \sigma_{-i/2}(x_k) - \sigma_{i/2}(x_k^*) x_k \right) f_0(t)\, dt, \tag{1.7}$$

where $f_0$ is the function given in (1.1). Moreover, under the condition (1.5), the self-adjoint operator $H$ can be written as

$$H = \sum_{k=1}^{n} H_k,$$

where each $H_k$, $k = 1, 2, \cdots, n$, is the Dirichlet operator associated to the Dirichlet form constructed in [23] with $x = x_k$ and $f = f_0$. See Theorem 2.2 for details. Thus conditions (1.5) and (1.6) are sufficient conditions for $H = H^*$.

In Section 5, we give a brief discussion on necessary and sufficient conditions for $H = H^*$ and show that, if $\xi_0$ defines a tracial state: $\langle \xi_0, AB\xi_0 \rangle = \langle \xi_0, BA\xi_0 \rangle$, $\forall A, B \in \mathcal{M}$, then the conditions (1.5) and (1.6) are also necessary conditions for $H = H^*$. Thus we believe that the conditions (1.5) and (1.6) are very close to necessary conditions for $H = H^*$ for any non-tracial $\xi_0$.

We organize the paper as follows: In Section 2, we introduce notation, definitions and necessary terminologies in the theory of noncommutative Dirichlet forms in the sense of Cipriani[13]. We then list our main results(Theorem 2.1, Proposition 2.1 and Theorem 2.2). We prove Theorem 2.1 in Section 3, and Proposition 2.1 and Theorem 2.2 in Section 4 respectively. In Section 5, we give a brief discussion on the necessary and sufficient conditions for $H = H^*$, and on the map $L$ on $\mathcal{M}$ associated to a Dirichlet operator $H$ for a general admissible function.

# 2 Notation, Definitions and Main Results

In this section, we first first introduce necessary terminologies in the theory of Dirichlet forms and Markovian semigroups on standard form of von Neumann algebras[13] and then list our main results.

Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$. Let $\xi_0 \in \mathcal{H}$ be a cyclic and separating vector for $\mathcal{M}$. We use $\Delta$ and $J$ to denote respectively, the modular operator and the modular conjugation associated with the pair $(\mathcal{M},\, \xi_0)$[10]. The associated modular automorphism group is denoted by $\sigma_t$: $\sigma_t(A) = \Delta^{it} A \Delta^{-it}, \forall A \in \mathcal{M},\, t \in \mathbb{R}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $*$-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$.

The natural positive cone $\mathcal{P}$ associated with the pair $(\mathcal{M},\, \xi_0)$ is the closure of the set

$$\{Aj(A)\xi_0 : A \in \mathcal{M}\}.$$

By a general result, the closed convex cone $\mathcal{P}$ can be obtained by the closure of the set

$$\{\Delta^{1/4} AA^* \xi_0 : A \in \mathcal{M}\}$$

and this cone $\mathcal{P}$ is self-dual in the sense that

$$\Big\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0,\ \forall \eta \in \mathcal{P}\Big\} = \mathcal{P}.$$

For the details we refer [5] and Section 2.5 of [10].

The form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with the pair $(\mathcal{M}, \xi_0)$. We shall use the fact that $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J = \Big\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R},\ \forall \eta \in \mathcal{P}\Big\}$, whose elements are called *J-real*: $\mathcal{H} = \mathcal{H}^J \bigoplus i\mathcal{H}^J$. The cone $\mathcal{P}$ gives rise to a structure of ordered Hilbert space on $\mathcal{H}^J$ (denoted by $\leq$) and to an anti-unitary involution $J$ on $\mathcal{H}$, which preserves $\mathcal{P}$ and $\mathcal{H}^J$: $J(\xi + i\eta) = \xi - i\eta$, $\forall \xi, \eta \in \mathcal{H}^J$. Also note that any $J$-real element $\xi \in \mathcal{H}^J$ can be decomposed uniquely as a difference of two mutually orthogonal, positive elements, called the positive and negative part of $\xi$, respectively : $\xi = \xi_+ - \xi_-$, $\xi_+, \xi_- \in \mathcal{P}$ and $\langle \xi_+, \xi_- \rangle = 0$. The order interval $\{\eta \in \mathcal{H} : 0 \leq \eta \leq \xi_0\}$ will be denoted by $[0, \xi_0]$. This is a closed convex subset of $\mathcal{H}$, and we shall denote the nearest point projection onto $[0, \xi_0]$ by $\eta \mapsto \eta_I$.

A bounded operator $A$ on $\mathcal{H}$ is called *J-real* if $AJ = JA$ and *positive preserving* if $A\mathcal{P} \subset \mathcal{P}$. The semigroup $\{T_t\}_{t \geq 0}$ is said to be *J-real* if $T_t$ is $J$-real for any $t \geq 0$ and it is called *positive preserving* if $T_t$ is positive preserving for any $t \geq 0$. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* (with respect to $\xi_0$) if $0 \leq \xi \leq \xi_0$ implies $0 \leq A\xi \leq \xi_0$. $A$ is called *Markovian* if it is sub-Markovian and also $A\xi_0 = \xi_0$. A semigroup $\{T_t\}_{t \geq 0}$ is said to be *sub-Markovian* (with respect to $\xi_0$) if $T_t$ is sub-Markovian for every $t \geq 0$. The semigroup $\{T_t\}_{t \geq 0}$ is called *Markovian* if $T_t$ is Markovian for every $t \geq 0$.

Next, we consider a sesquilinear form on some linear manifold of $\mathcal{H}$ : $\mathcal{E}(\cdot,\cdot) : D(\mathcal{E})\times D(\mathcal{E}) \to \mathbb{C}$. We also consider the associated quadratic form: $\mathcal{E}[\cdot] : D(\mathcal{E}) \to \mathbb{C}$, $\mathcal{E}[\xi] := \mathcal{E}(\xi,\xi)$. A real valued quadratic form $\mathcal{E}[\cdot]$ is said to be *semi-bounded* if $\inf\{\mathcal{E}[\xi] : \xi \in D(\mathcal{E}),\ ||\xi|| = 1\} = -b > -\infty$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be *J-real* if $JD(\mathcal{E}) \subset D(\mathcal{E})$ and $\mathcal{E}[J\xi] = \overline{\mathcal{E}[\xi]}$ for any $\xi \in D(\mathcal{E})$. For a given semi-bounded quadratic form $\mathcal{E}$, one considers the inner product given by $\langle\xi,\eta\rangle_\lambda := \mathcal{E}(\xi,\eta) + \lambda\langle\xi,\eta\rangle$, for $\lambda > b$. The form $\mathcal{E}$ is *closed* if $D(\mathcal{E})$ is a Hilbert space for some of the above norms. The form $\mathcal{E}$ is called *closable* if it admits a closed extension.

Associated to a semi-bounded closed form $\mathcal{E}$, there are a self-adjoint operator $(H, D(H))$ and a strongly continuous, symmetric semigroup $\{T_t\}_{t\geq 0}$. Each of the above objects determines uniquely the others according to well known relations (see [26] and Section 3.1 of [10]).

A $J$-real, real-valued, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called *Markovian* with respect to $\xi_0 \in \mathcal{P}$ if

$$\eta \in D(\mathcal{E})^J \text{ implies } \eta_I \in D(\mathcal{E}) \text{ and } \mathcal{E}[\eta_I] \leq \mathcal{E}[\eta],$$

where $D(\mathcal{E})^J := D(\mathcal{E}) \cap \mathcal{H}^J$. A closed Markovian form is called a *Dirichlet form.*

Next, we collect main results of [13]. Let $(\mathcal{E}, D(\mathcal{E}))$ be a $J$-real, real valued, densely defined closed form. Assume that the following properties hold:

$$\begin{aligned}
&(a)\quad \xi_0 \in D(\mathcal{E}), \\
&(b)\quad \mathcal{E}(\xi,\xi) \geq 0 \text{ for } \xi \in D(\mathcal{E}), \\
&(c)\quad \xi \in D(\mathcal{E})^J \ \text{ implies } \ \xi_\pm \in D(\mathcal{E}) \ \text{ and } \mathcal{E}(\xi_+,\xi_-) \leq 0.
\end{aligned} \tag{2.1}$$

Then $\mathcal{E}$ is a Dirichlet form if and only if $\mathcal{E}(\xi,\xi_0) \geq 0$ for all $\xi \in D(\mathcal{E}) \cap \mathcal{P}$. The above result follows from Proposition 4.5 (b) and Proposition 4.10 (ii) of [13].

The following is Theorem 4.11 of [13] : Let $\{T_t\}_{t\geq 0}$ be a $J$-real, strongly continuous, symmetric semigroup on $\mathcal{H}$ and let $(\mathcal{E}, D(\mathcal{E}))$ be the associated densely defined $J$-real, real valued quadratic form. Then the followings are equivalent.

$$\begin{aligned}
&(a)\quad \{T_t\}_{t\geq 0} \ \text{ is sub-Markovian.} \\
&(b)\quad (\mathcal{E}, D(\mathcal{E})) \ \text{ is a Dirichlet form .}
\end{aligned} \tag{2.2}$$

We refer the reader to [13] for the details.

Next, we give an extended version of the general construction method developed in [23]. For any $\lambda > 0$, denote by $I_\lambda$ the closed strip given by

$$I_\lambda = \{z : z \in \mathbb{C}, |Im\, z| \leq \lambda\}. \tag{2.3}$$

Let us introduce the notion of admissible functions [23].

**Definition 2.1** *An analytic function $f : D \to \mathbb{C}$ on a domain $D$ containing the strip $I_{1/4}$ is said to be admissible if the following properties hold:*

*(a)* $f(t) \geq 0$ *for* $\forall t \in \mathbb{R}$,

*(b)* $f(t+i/4) + f(t-i/4) \geq 0$ *for* $\forall t \in \mathbb{R}$,

*(c) there exist* $M > 0$ *and* $p > 1$ *such that the bound*

$$|f(t+is)| \leq M(1+|t|)^{-p}$$

*holds uniformly in* $s \in [-1/4, 1/4]$.

We remark that there exist a non-trivial admissible function[23, Lemma 3.1].

Next, we consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ given by

$$f_0(t) = 2(e^{2\pi t} + e^{-2\pi t})^{-1}. \tag{2.4}$$

The function $f_0$ will play important roles in the sequels. Using the residue integration method, it is easy to check that

$$2\int (e^{2\pi t} + e^{-2\pi t})^{-1} e^{ikt}\, dt = (e^{k/4} + e^{-k/4})^{-1}. \tag{2.5}$$

See also the expression in P. 94 of [10]. One can see that $f_0$ has an analytic extension, denoted by $f_0$ again, to the interior of $I_{1/4}$. On the boundary of $I_{1/4}$, it defines a distribution, and satisfies the equality

$$f(t+i/4) + f(t-i/4) = \delta(t)$$

in the sense of distribution. Thus even if $f_0$ is not an admissible function, it is almost admissible.

For any $\lambda > 0$, denote by $\mathcal{M}_\lambda$ the dense subset of $\mathcal{M}$ consisting of every $\sigma_t$-analytic element with a domain containing $I_\lambda$. By Proposition 2.5.21 of [10], any $A \in \mathcal{M}_\lambda$ is strongly analytic. In the following, the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is conjugate linear in the first and linear in the second variable. For given $x \in \mathcal{M}_{1/4}$ and an admissible function $f$ or else $f = f_0$, define a sesquilinear form $\mathcal{E} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ by

$$\begin{aligned}
&\mathcal{E}(\eta, \xi) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.6)\\
&= \int \Big\langle \big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))\big)\eta, \big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))\big)\xi \Big\rangle f(t)dt \\
&\quad + \int \Big\langle \big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))\big)\eta, \big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))\big)\xi \Big\rangle f(t)dt
\end{aligned}$$

The form is positive and bounded. The self-adjoint operator $H$ associated to the form is given by

$$\begin{aligned}
H \;=\; &\int \big(\sigma_{t+i/4}(x^*) - j(\sigma_{t+i/4}(x))\big)\big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))\big) f(t)\, dt \qquad (2.7)\\
&\int \big(\sigma_{t+i/4}(x) - j(\sigma_{t+i/4}(x^*))\big)\big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))\big) f(t)\, dt
\end{aligned}$$

The following result is an extended version of Theorem 3.1 of [23].

**Theorem 2.1** *Let $f$ be either an admissible function or else $f = f_0$ and $x \in \mathcal{M}_{1/4}$. Let $(\mathcal{E}, \mathcal{H})$ be the quadratic form associated to the sesquilinear form defined as in (2.6) : $\mathcal{E}[\xi] = \mathcal{E}(\xi, \xi)$. Let $H$ be the self-adjoint operator associated with $(\mathcal{E}, \mathcal{H})$. Then the following properties hold:*

*(a) $H\xi_0 = 0$,*

*(b) $\mathcal{E}$ is $J$-real*

*(c) $\mathcal{E}(\xi_+, \xi_-) \leq 0 \quad \forall \xi \in \mathcal{H}^J$.*

*Furthermore the form $(\mathcal{E}, \mathcal{H})$ is a Dirichlet form.*

The proof of the theorem will be given in the next section. It may be worth to compare Theorem 3.1 of [23] and Theorem 2.1 in the above, and give a comment on possible extensions of Theorem 2.1.

**Remark 2.1** *(a) In [23], the properties (a),(b) and (c) in Theorem 2.1 were proved under assumptions that $x = x^* \in \mathcal{M}$, $f$ is admissible and that there exist a constant $M > 0$ such that the bound*

$$\sup_{s \in [-1/4, 1/4]} \|\sigma_{t+is}(x)\| \leq M$$

*holds uniformly in $t \in \mathbb{R}$. Notice that if one writes*

$$x_1 := \frac{1}{\sqrt{2}}(x + x^*), \quad x_2 := \frac{i}{\sqrt{2}}(x - x^*), \tag{2.8}$$

*then $\mathcal{E}(\eta, \xi)$ can be written as*

$$\mathcal{E}(\eta, \xi) = \frac{1}{2}\{\mathcal{E}_1(\eta, \xi) + \mathcal{E}_2(\eta, \xi)\}, \tag{2.9}$$

*where $\mathcal{E}_j(\eta, \xi)$, $j = 1, 2$ is the form corresponding to $x_j$, $i = 1, 2$, respectively. Thus one may assume that $x$ is self-adjoint. On the other hand, the above bound need to prove the property (c) by Cauchy's integral theorem. We will give another proof of the property (c) which do not use the above bound.*

*(b) In applications, one may choose $\{x_k\}_{k=1}^{\infty} \in \mathcal{M}_{1/4}$ which generates $\mathcal{M}$, and $f$, where $f$ is an admissible function or else $f = f_0$. For each $k \in \mathbb{N}$, let $(\mathcal{E}_k, \mathcal{H})$ be the Dirichlet form obtained from (2.6) with $x = x_k$. Let $(\mathcal{E}, D(\mathcal{E}))$ be the sesquilinear form defined by*

$$\begin{aligned} D(\mathcal{E}) &= \{\xi \in \mathcal{H} : \sum_{k=1}^{\infty} \mathcal{E}_k(\xi, \xi) < \infty\}, \\ \mathcal{E}(\eta, \xi) &= \sum_{k=1}^{\infty} \mathcal{E}_k(\eta, \xi), \quad \eta, \xi \in D(\mathcal{E}). \end{aligned}$$

*If $D(\mathcal{E})$ is dense in $\mathcal{H}$, then $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form [13, Theorem 5.2]. The above method has been used in [8, 9]. Also it may be possible to extend Theorem 2.1 to the case in which $x$ is an unbounded operator affiliated with $\mathcal{M}$ [8, 7].*

As discussed in Introduction, we will consider Lindblad type generators of conservative q.d.s. on $\mathcal{M}$ and their symmetric embeddings. Recall that $\mathcal{M}_{1/2}$ denotes the dense subset of $\mathcal{M}$ consisting of every $\sigma_t$-analytic element with a domain containing the strip $I_{1/2}$. For given $y \in \mathcal{M}_{1/2}$ and $Q = Q^* \in \mathcal{M}_{1/2}$, we first consider the following Lindblad type generator of a q.d.s. on $\mathcal{M}$ :

$$\begin{aligned} L &: \mathcal{M} \to \mathcal{M} \\ L(A) &= y^*yA - 2y^*Ay + Ay^*y + i[Q, A], \quad A \in \mathcal{M}, \end{aligned} \tag{2.10}$$

where $[A, B] := AB - BA$, $A, B \in \mathcal{M}$. Consider the following symmetric embedding [13] :

$$\begin{aligned} i_0 &: \mathcal{M} \to \mathcal{H} \\ i_0(A) &= \Delta^{1/4} A\xi_0, \quad A \in \mathcal{M}, \end{aligned} \tag{2.11}$$

and define the operator $H$ on $\mathcal{H}$ by

$$H\Delta^{1/4} A\xi_0 = \Delta^{1/4} L(A)\xi_0, \quad A \in \mathcal{M}. \tag{2.12}$$

It is easy to see that $H$ is self-adjoint if and only if $L$ satisfied the following *KMS symmetry* [13, 16] : For any $A, B \in \mathcal{M}_{1/4}$

$$\langle \sigma_{-i/4}(L(A))\xi_0, \sigma_{-i/4}(B)\xi_0 \rangle = \langle \sigma_{-i/4}(A)\xi_0, \sigma_{-i/4}(L(B))\xi_0 \rangle.$$

According to [13, Proposition 2.4 and Theorem 2.12], the map $L$ generates a (weak* continuous) KMS symmetric, conservative q.d.s. on $\mathcal{M}$ if and only if $H$ generates a (strongly continuous) symmetric Markovian semigroup on $\mathcal{H}$. The following result can be considered as a structure theorem for Dirichlet forms on the standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ associated to the pair $(\mathcal{M}, \xi_0)$.

**Proposition 2.1** *For given $y \in \mathcal{M}_{1/2}$ and $Q = Q^* \in \mathcal{M}_{1/2}$, let $L : \mathcal{M} \to \mathcal{M}$ be given as (2.10). Put $x := \sigma_{i/4}(y)$. Assume that the relation*

$$xj(x) = x^*j(x^*) \tag{2.13}$$

*holds. Let $H$ be the operator on $\mathcal{H}$ defined as (2.12). Then $H$ is self-adjoint if and only if $Q$ is given by*

$$Q = i \int \big(\sigma_t(x^*)\sigma_{t-i/2}(x) - \sigma_{t+i/2}(x^*)\sigma_t(x)\big)\, f_0(t)\, dt, \tag{2.14}$$

*where $f_0$ is the function given in (2.4). Moreover the self-adjoint operator $H$ can be expresses in the form of the Dirichlet operator given as (2.7) with $f = f_0$.*

The proof of the above Proposition will be produced in Section 4. The following result is a generalization of Proposition 2.1.

**Theorem 2.2** *Let $y_k$, $k = 1, 2, \cdots, n$, be elements of $\mathcal{M}_{1/2}$, and $Q = Q^* \in \mathcal{M}_{1/2}$. Let $L$ be the map of $\mathcal{M}$ into itself given by*

$$L(A) = \sum_{k=1}^{n} (y_k^* A y_k - 2y_l^* A y_k + A y_k^* y_k) + i[Q, A], \quad A \in \mathcal{M}. \tag{2.15}$$

*Put $x_k := \sigma_{i/4}(y)$, $k = 1, 2, \cdots, n$. Assume that the relation*

$$\sum_{k=1}^{n} x_k j(x_k) = \sum_{k=1}^{n} x_k^* j(x_k^*) \tag{2.16}$$

*holds. Let $H$ be the operator on $\mathcal{H}$ defined as (2.12). Then $H$ is self-adjoint if and only if $Q$ is given by*

$$Q = \sum_{k=1}^{n} Q_k \tag{2.17}$$

*where each $Q_k$, $k = 1, 2, \cdots, n$, is given as in (2.14) with $x$ replaced by $x_k$. Moreover the self-adjoint operator can be written as*

$$H = \sum_{k=1}^{n} H_k,$$

*where each $H_k$, $k = 1, 2, \cdots, n$, is given as (2.7) with $x = x_k$ and $f = f_0$.*

We give comments on the conditions (2.16) and its consequences and possible extension of Theorem 2.2 to the general Lindblad type generator given in (1.2):

**Remark 2.2** *(a) The conditions (2.16) and (2.17) are sufficient conditions for the map $L$ given in (2.15) to be KMS symmetric, or equivalently the operator $H$ induced by $L$ to be self-adjoint. If $\xi_0$ is tracial : $\langle \xi_0, AB\xi_0 \rangle = \langle \xi_0, BA\xi_0 \rangle$, $\forall A, B \in \mathcal{M}$, then the conditions (2.16) and (2.17) are also necessary conditions for $L$ to be (KMS) symmetric. See Section 5.*

*(b) This condition (2.16) is equivalent to the following condition:*

$$\sum_{k=1}^{n} \sigma_{i/4}(x_k^*) A \sigma_{-i/4}(x_k) = \sum_{k=1}^{n} \sigma_{i/4}(x_k) A \sigma_{-i/4}(x_k^*), \quad \forall A \in \mathcal{M}. \tag{2.18}$$

*See Lemma 4.1 (b). In terms of $x_k$'s, $L(A)$ in (2.15) is given by*

$$\begin{aligned} L(A) \;\; = \;\; & \sum_{k=1}^{n} \{\sigma_{i/4}(x_k^*)\sigma_{-i/4}(x_k)A - 2\sigma_{i/4}(x_k^*)A\sigma_{-i/4}(x_k) + A\sigma_{i/4}(x_k^*)\sigma_{-i/4}(x_k)\} \\ & +i[Q, A]. \end{aligned}$$

*For given* $\{x_1, x_2, \cdots, x_n\} \subset \mathcal{M}_{1/4}$*, let* $\{\widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_{2n}\} \subset \mathcal{M}_{1/4}$ *be the family of self-adjoint elements defined by*

$$\widetilde{x}_{2k} = \frac{1}{\sqrt{2}}(x_k + x_k^*), \quad \widetilde{x}_{2k-1} = \frac{i}{\sqrt{2}}(x_k - x_k^*), \;\; k = 1, 2, \cdots, n.$$

*Then, under the condition (2.18), the KMS symmetric map* $L$ *can be written as*

$$L(A) = \frac{1}{2}\sum_{k=1}^{2n} L_k(A), \tag{2.19}$$

*where for* $k = 1, 2, \cdots, 2n$,

$$\begin{aligned} L_k(A) \;\; = \;\; & \sigma_{i/4}(\widetilde{x}_k)\sigma_{-i/4}(\widetilde{x}_k)A - 2\sigma_{i/4}(\widetilde{x}_k)A\sigma_{-i/4}(\widetilde{x}_k) + A\sigma_{i/4}(\widetilde{x}_k)\sigma_{-i/4}(\widetilde{x}_k) \\ & + i[Q_k, A], \end{aligned} \tag{2.20}$$

*where* $Q_k$ *is given by (2.14) with* $x = \widetilde{x}_k$.

*(c) For any family* $\{x_k\}_{k=1}^{\infty} \subset \mathcal{M}_{1/4}$ *of self-adjoint elements, consider the following Lindblad type generator*

$$L(A) = \sum_{k=1}^{\infty} L_k(A), \quad A \in \mathcal{M}, \tag{2.21}$$

*where each* $L_k(A)$, $k \in \mathbb{N}$, *is given by (2.20) with* $x_k$ *replacing* $\widetilde{x}_k$. *Since each* $L_k$, $k \in \mathbb{N}$, *is KMS symmetric, the map* $L$ *given above is formally KMS symmetric, and the operator* $H$ *induced by* $L$ *is given by*

$$H = \sum_{k=1}^{\infty} H_k \tag{2.22}$$

*where each* $H_k$ *is the Dirichlet operator given by (2.7) with* $f = f_0$ *and* $x_k$ *replacing* $x$. *The expressiions in (2.21) and (2.22) are still formal. In order to give rigorous meanings to the expressions, one has to give dense domains* $D(L)$ *and* $D(H)$ *such that the right hand sides of (2.21) and (2.22) are well-defined. Since* $Q_k$ *and* $H_k$ *,* $k \in \mathbb{N}$, *are given by integral forms as in (2.14) and (2.7) respectively, the task would not so simple. It would be very nice if one can give a sufficient condition on* $\{x_k\}_{k=1}^{\infty}$, *which is easy to verify for concrete models, such that the right hand sides of (2.21) and (2.22) converge in a appropriate sense. See Remark 2.1 (b).*

# 3 Proof of Theorem 2.1

Before proving Theorem 2.1, let us introduce linear maps on $\mathcal{L}(\mathcal{H})$ which will be used frequently in the sequels. For any $\lambda > 0$, denote by $\mathcal{L}_\lambda(\mathcal{H})$ the dense subset of

$\mathcal{L}(\mathcal{H})$ consisting of every $\sigma_t$-analytic element of $\mathcal{L}(\mathcal{H})$ with a domain containing the strip $I_\lambda$. Let $D_{1/4}$ and $D_{-1/4}$ be the linear maps on $\mathcal{L}(\mathcal{H})$ defined by

$$\begin{aligned} D(D_{1/4}) &= \mathcal{L}_{1/4}(\mathcal{H}), \\ D_{1/4}(A) &= \sigma_{-i/4}(A), \quad A \in \mathcal{L}_{1/4}(\mathcal{H}), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} D(D_{-1/4}) &= \mathcal{L}_{1/4}(\mathcal{H}), \\ D_{-1/4}(A) &= \sigma_{i/4}(A), \quad A \in \mathcal{L}_{1/4}(\mathcal{H}). \end{aligned} \tag{3.2}$$

Put

$$\begin{aligned} T &:= D_{1/4} + D_{-1/4}, \\ S &:= D_{1/4} - D_{-1/4}. \end{aligned} \tag{3.3}$$

Let $I_0$ be the linear map defined by

$$\begin{aligned} D(I_0) &= \mathcal{L}(\mathcal{H}), \\ I_0(A) &= \int \sigma_t(A) f_0(t) dt, \quad A \in \mathcal{L}(\mathcal{H}), \end{aligned} \tag{3.4}$$

where $f_0$ is the function given in (2.4).

We have the following result which will be used in the proofs of the results in Section 2.

**Lemma 3.1** *The relations*

$$T I_0(A) = I_0 T(A) = A$$

*hold for any* $A \in \mathcal{L}_{1/4}(\mathcal{H})$. *That is,* $T$ *is invertible and* $T^{-1} = I_0$.

*Proof:* The proof of the above lemma is essentially contained in the proof of Theorem 2.5.14(Tomita-Takesaki theorem) of [10]. Since the method of the proof will be used in the proof of Theorem 2.1, we produce the proof.

As a relation between bilinear forms on $D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$, one has

$$T(A) = \Delta^{1/4} A \Delta^{-1/4} + \Delta^{-1/4} A \Delta^{1/4}, \quad A \in \mathcal{L}(\mathcal{H}).$$

Now take $\eta, \xi \in D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$. Then it follows that for any $A \in \mathcal{L}_{1/4}(\mathcal{H})$

$$\begin{aligned} \langle \eta, T(I_0(A))\xi \rangle &= \langle \Delta^{1/4}\eta, I_0(A)\Delta^{-1/4}\xi \rangle + \langle \Delta^{-1/4}\eta, I_0(A)\Delta^{1/4}\xi \rangle \\ &= \int \left( \langle \Delta^{-it+1/4}\eta, A\Delta^{-it-1/4}\xi \rangle + \langle \Delta^{-it-1/4}\eta, A\Delta^{-it+1/4}\xi \rangle \right) f_0(t)\, dt. \end{aligned} \tag{3.5}$$

Denote by $h$ the generator of $\Delta^{it}$ : $h := \log(\Delta)$. Using the spectral decomposition of $h$:

$$h = \int \mu \, dE(\mu),$$

one obtains that

$$\begin{aligned}
&\langle \eta, T(I_0(A))\xi\rangle \\
&\quad = \int f_0(t) \left\{ \int d^2\langle E(\mu)\eta, AE(\rho)\xi\rangle \left(e^{(\mu-\rho)/4} + e^{-(\mu-\rho)/4}\right) e^{i(\mu-\rho)t} \right\} dt.
\end{aligned}$$

The domain restrictions on $\eta$ and $\xi$ allow interchange of the order of integrations and one has

$$\begin{aligned}
&\langle \eta, T(I_0(A))\xi\rangle \qquad (3.6)\\
&\quad = \int d^2\langle E(\mu)\eta, AE(\rho)\xi\rangle \left(e^{(\mu-\rho)/4} + e^{-(\mu-\rho)/4}\right) \int f_0(t) e^{i(\mu-\rho)t} \, dt \\
&\quad = \int d^2\langle E(\mu)\eta, AE(\rho)\xi\rangle \\
&\quad = \langle \eta, A\xi\rangle,
\end{aligned}$$

where the first step uses the Fourier relation in (2.5). Thus as a relation between bilinear forms in $D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$, we have

$$T(I_0(A)) = A.$$

For any $A \in \mathcal{L}_{1/4}(\mathcal{H})$, $I_0(A) \in \mathcal{L}_{1/4}(\mathcal{H})$ and $T(I_0(A)) \in \mathcal{L}(\mathcal{H})$. Since $D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$ is dense in $\mathcal{H}$. the above equality holds as a relation between bounded operators. It follows from the definitions of $T$ and $I_0$ in (3.3) and (3.4) respectively that $T$ and $I_0$ commute on $\mathcal{L}_{1/4}(\mathcal{H})$. This completes the proof of the lemma. □

We now turn to the proof of Theorem 2.1.

*Proof of Theorem 2.1.* The proof of the properties (a) and (b) is same as that in the proof of Theorem 2.1 of [23]. See also the proof of Theorem 2.1 of [8].

We prove that property (c). By the expression of $\mathcal{E}(\eta, \xi)$ in (2.6), $\mathcal{E}(\xi_+, \xi_-)$ can be written as

$$\begin{aligned}
\mathcal{E}(\xi_+, \xi_-) &= \mathcal{E}^{(1)}(\xi_+, \xi_-) + \mathcal{E}^{(2)}(\xi_+, \xi_-) \\
&= (\mathrm{I}^{(1)} + \mathrm{II}^{(1)}) + (\mathrm{I}^{(2)} + \mathrm{II}^{(2)}), \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}^{(1)} &= \int \left( \langle \sigma_{t-i/4}(x)\xi_+, \sigma_{t-i/4}(x)\xi_-\rangle + \langle \sigma_{t-i/4}(x^*)\xi_-, \sigma_{t-i/4}(x^*)\xi_+\rangle \right) f(t) \, dt \qquad (3.8)\\
\mathrm{II}^{(1)} &= -\int \left( \langle \sigma_{t-i/4}(x)\xi_+, j(\sigma_{t-i/4}(x^*))\xi_-\rangle + \langle j(\sigma_{t-i/4}(x^*))\xi_+, \sigma_{t-i/4}(x)\xi_-\rangle \right) f(t) \, dt,
\end{aligned}$$

and $\mathrm{I}^{(2)}$ and $\mathrm{II}^{(2)}$ are obtained from $\mathrm{I}^{(1)}$ and $\mathrm{II}^{(1)}$, respectively, replacing $x$ by $x^*$ in the above. Here we have used the fact that $\langle J\eta, J\xi\rangle = \langle \xi, \eta\rangle$ in $\mathrm{I}^{(1)}$.

As a consequence of Theorem 4(7) of [5], $\mathcal{M}\xi_+ \perp \mathcal{M}\xi_-$, which implies $\mathrm{I}^{(1)} = 0$ and $\mathrm{I}^{(2)} = 0$. Next, we consider $\mathrm{II}^{(1)}$. It can be checked that $\sigma_{t-is}(x)^* = \sigma_{t+is}(x^*)$ and $j(\sigma_{t+is}(x)) = \sigma_{t-is}(j(x))$ for any $x \in \mathcal{M}_{1/2}$ and $s \in [-1/4, 1/4]$, and so

$$\begin{aligned}
\langle \sigma_{t-i/4}(x)\xi_+, j(\sigma_{t-i/4}(x^*))\xi_-\rangle &= \langle \xi_+, \sigma_{t+i/4}(x^* j(x^*))\xi_-\rangle, \\
\langle j(\sigma_{t-i/4}(x^*))\xi_+, \sigma_{t-i/4}(x)\xi_-\rangle &= \langle \xi_+, \sigma_{t-i/4}(xj(x))\xi_-\rangle.
\end{aligned}$$

It follows from the definition of $\mathrm{II}^{(1)}$ in (3.8) that

$$\begin{aligned}
\mathrm{II}^{(1)} &= -\int \langle \xi_+, \sigma_{t+i/4}(x^* j(x^*))\xi_-\rangle f(t)\, dt \\
&\quad - \int \langle \xi_+, \sigma_{t-i/4}(xj(x))\xi_-\rangle f(t)\, dt
\end{aligned}$$

Replacing $x$ by $x^*$ in the above, we obtain the expression of $\mathrm{II}^{(2)}$. Thus we get

$$\begin{aligned}
\mathrm{II} &= \mathrm{II}^{(1)} + \mathrm{II}^{(2)} \\
&= -\int \langle \xi_+, T\left(\sigma_t(xj(x) + x^* j(x^*))\right)\xi_-\rangle f(t)\, dt. \qquad (3.9)
\end{aligned}$$

We first consider the case for $f = f_0$. From the definition of $I_0$ in (3.4) and Lemma 3.1, we have

$$\begin{aligned}
\mathrm{II} &= -\langle \xi_+, T(I_0(xj(x) + x^* j(x^*)))\xi_-\rangle \\
&= -\langle \xi_+, (xj(x) + x^* j(x^*))\xi_-\rangle \\
&\leq 0.
\end{aligned}$$

Here we have used the fact that $Aj(A)\xi_- \in \mathcal{P}$ for any $A \in \mathcal{M}$.

Next we consider any admissible function $f$. For any $\eta, \zeta \in D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$, consider the following expression:

$$B(\eta, \zeta) := -\int \langle \eta, T(\sigma_t(xj(x) + x^* j(x^*)))\zeta\rangle f(t)\, dt. \qquad (3.10)$$

Employing the method similar to that used to obtain the first relation of (3.6) from (3.5), we have

$$\begin{aligned}
&B(\eta, \zeta) \\
&= -\int d^2 \langle E(\mu)\eta, (xj(x) + x^* j(x^*))E(\rho)\zeta\rangle (e^{(\mu-\rho)/4} + e^{-(\mu-\rho)/4}) \\
&\qquad \cdot \int f(t) e^{i(\mu-\rho)}\, dt.
\end{aligned}$$

We now use the bound (c) in Definition 2.1 and Cauchy's integral theorem to conclude that

$$
\begin{aligned}
&(e^{(\mu-\rho)/4} + e^{-(\mu-\rho)/4}) \int f(t) e^{i(\mu-\rho)}\, dt \\
&\qquad = \int \left(f(t-i/4) + f(t+i/4)\right) e^{i(\mu-\rho)t}\, dt,
\end{aligned}
$$

and so

$$
\begin{aligned}
&B(\eta,\zeta) \\
&= -\int d^2 \langle E(\mu)\eta, (xj(x) + x^* j(x^*)) E(\rho)\zeta \rangle \left(f(t-i/4) + f(t+i/4)\right) e^{i(\mu-\rho)t}\, dt \\
&= -\int \langle \eta, \sigma_t(xj(x) + x^* j(x^*))\zeta \rangle \left(f(t-i/4) + f(t+i/4)\right)\, dt.
\end{aligned}
$$

Thus as a relation between bilinear form on $D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$, one has

$$
\begin{aligned}
&\int T(\sigma_t(xj(x) + x^* j(x^*))) f(t)\, dt \\
&\qquad = \int \sigma_t(xj(x) + x^* j(x^*))(f(t+i/4) + t(t-i/4))\, dt.
\end{aligned}
\tag{3.11}
$$

Since the linear operators in the above are well-defined bounded operators by the bound (c) and the fact that $xj(x) + x^* j(x^*) \in \mathcal{L}_{1/4}(\mathcal{H})$ and since $D(\Delta^{1/4}) \cap D(\Delta^{-1/4})$ is dense in $\mathcal{H}$, the relation (3.11) holds as a relation between bounded operators. It follows from (3.9) and (3.11) that

$$
\begin{aligned}
\mathrm{II} &= -\int \langle \xi_+, \sigma_t(xj(x) + x^* j(x^*))\xi_- \rangle (f(t+i/4) + t(t-i/4))\, dt \\
&\leq 0.
\end{aligned}
$$

Here we have used the property (c) in the Definition 2.1 and the fact that $\sigma_t(Aj(A))\xi_- = \sigma_t(A) j(\sigma_t(A))\xi_- \in \mathcal{P}$ for any $A \in \mathcal{M}$. This proved the property (c).

Clearly $\mathcal{E}(\xi,\xi) \geq 0$, $\forall \xi \in \mathcal{H}$. Thus the properties (a), (b) and (c) in (2.1) hold. Since $\left(\sigma_{-i/4}(B) - j(\sigma_{-i/4}(B^*))\right)\xi_0 = 0$ for any $B \in \mathcal{M}_{1/4}$, $\mathcal{E}(\xi,\xi_0) = 0$ for any $\xi \in \mathcal{P}$. Thus $\mathcal{E}$ is a Dirichlet form by Proposition 4.5(b) and Proposition 4.10 (ii) of [13]. □

# 4 Proofs of Proposition 2.1 and Theorem 2.2

In this section we prove Proposition 2.1 and Theorem 2.2. We first establish relations equivalent to (2.13) and (2.16) respectively:

**Lemma 4.1** *(a) For a given $x \in \mathcal{M}_{1/4}$, the relation (2.13) holds if and only if*

$$\sigma_{i/4}(x) A \sigma_{-i/4}(x^*) = \sigma_{i/4}(x^*) A \sigma_{-i/4}(x)$$

*for any $A \in \mathcal{M}$.*

*(b) For given $\{x_1, x_2, \cdots, x_n\} \subset \mathcal{M}_{1/4}$, the relation (2.16) holds if and only if*

$$\sum_{k=1}^{n} \sigma_{i/4}(x_k) A \sigma_{-i/4}(x_k^*) = \sum_{k=1}^{n} \sigma_{i/4}(x_k^*) A \sigma_{-i/4}(x_k)$$

*for any $A \in \mathcal{M}$.*

*Proof:* (a) By acting $D_{-1/4}$ on the both sides of (2.13), it can be checked that the condition (2.13) is equivalent to the following condition:

$$\sigma_{i/4}(x) j(\sigma_{-i/4}(x)) = \sigma_{i/4}(x^*) j(\sigma_{-i/4}(x^*))$$

Since $j(\sigma_{-i/4}(B))\xi_0 = \sigma_{-i/4}(B^*)\xi_0$ for any $B \in \mathcal{M}_{1/4}$. one has that for any $A \in \mathcal{M}$

$$\begin{aligned}
\sigma_{i/4}(x) j(\sigma_{-i/4}(x)) A\xi_0 &= \sigma_{i/4}(x) A \sigma_{-i/4}(x^*)\xi_0, \\
\sigma_{i/4}(x^*) j(\sigma_{-i/4}(x^*)) A\xi_0 &= \sigma_{i/4}(x^*) A \sigma_{-i/4}(x)\xi_0.
\end{aligned}$$

Since $\mathcal{M}\xi_0$ is dense, (2.13) holds if and only if

$$\sigma_{i/4}(x) A \sigma_{-i/4}(x^*)\xi_0 = \sigma_{i/4}(x^*) A \sigma_{-i/4}(x)\xi_0.$$

for any $A \in \mathcal{M}$. Since $\xi_0$ is a separating vector, we proved (a).

(b) If one replaces $x$ and $x_k$ and sums over $k$ from 1 to $n$ in the above, the proof of the part (b) follows from that of the part (a). $\square$

We now turn to the proof of Proposition 2.1. Recall the definitions of the linear maps $D_{1/4}$, $D_{-1/4}, T, S$ and $I_0$ on $\mathcal{L}(\mathcal{H})$ defined as in (3.1) - (3.4).

*Proof of Proposition 2.1.* Notice that for any $y \in \mathcal{M}_{1/2}$ and $A \in \mathcal{L}_{1/2}(\mathcal{H})$ equalities

$$\begin{aligned}
\Delta^{1/2} A\xi_0 &= \sigma_{-i/2}(A)\xi_0, \\
y\xi_0 &= J\Delta^{1/2} y^*\xi_0 = j(\sigma_{-i/2}(y^*))\xi_0
\end{aligned} \tag{4.1}$$

hold. Let $L$ be given as in (2.10). A direct computation yields

$$\begin{aligned}
\Delta^{1/4} L(A)\xi_0 &= \Delta^{1/4}(y^*y - 2y^* j(\sigma_{-i/2}(y^*)) + j(\sigma_{-i/2}(y^*y)))A\xi_0 \\
&\quad + \Delta^{1/4}(iQ - ij(\sigma_{-i/2}(Q)))A\xi_0 \\
&= D_{1/4}(y^*y - 2y^* j(\sigma_{-i/2}(y^*)) - j(\sigma_{-i/2}(y^*y)))\sigma_{-i/4}(A)\xi_0 \\
&\quad + i\left(D_{1/4}(Q) - D_{1/4}(j(\sigma_{-i/2}(Q)))\right)\sigma_{-i/4}(A)\xi_0
\end{aligned}$$

Thus it follows from (2.12) and the above relation that

$$\begin{aligned} H &= D_{1/4}(y^*y - 2y^*j(\sigma_{-i/2}(y^*)) - j(\sigma_{-i/2}(y^*y))) \\ &\quad +iD_{1/4}(Q) - iD_{1/4}(j(\sigma_{-i/2}(Q))). \end{aligned}$$

Since $D_{1/4}(j(\sigma_{-i/2}(B))) = D_{-1/4}(j(B))$ for any $B \in \mathcal{M}_{1/2}$, $H$ can be written as

$$\begin{aligned} H &= D_{1/4}(y^*y) - 2\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*)) + D_{-1/4}(j(y^*y)) \\ &\quad +iD_{1/4}(Q) - iD_{-1/4}(j(Q)). \end{aligned} \tag{4.2}$$

Since $(D_{\pm 1/4}(B))^* = D_{\mp 1/4}(B^*)$ for any $B \in \mathcal{L}_{1/2}(\mathcal{H})$, one has

$$\begin{aligned} H^* &= D_{-1/4}(y^*y) - 2\sigma_{i/4}(y)j(\sigma_{i/4}(y)) + D_{1/4}(j(y^*y)) \\ &\quad -iD_{-1/4}(Q) + iD_{1/4}(j(Q)). \end{aligned} \tag{4.3}$$

Thus $H = H^*$ if and only if

$$\begin{aligned} &iT(Q) - iT(j(Q)) \\ &= -S(y^*y) + 2\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*)) - 2\sigma_{i/4}(y)j(\sigma_{i/4}(y)) + S(j(y^*y)). \end{aligned} \tag{4.4}$$

If the relation (2.13) holds, $H$ is self-adjoint if and only if

$$iT(Q) - iT(j(Q)) = -S(y^*y) + S(j(y^*y)),$$

and so by Lemma 3.1,

$$\begin{aligned} Q - j(Q) &= iI_0S(y^*y) - iI_0S(j(y^*y)) \\ &= I_0(iS(y^*y)) - j(I_0(iS(y^*y))). \end{aligned}$$

Here we have used the fact $S(j(y^*y)) = -j(S(y^*y))$. Thus $Q$ can be written as

$$Q = iI_0(S(y^*y)) + Q_c \tag{4.5}$$

for some $Q_c \in \mathcal{Z}(\mathcal{M})$, where $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$. Since $Q_c$ has no contributions in $L$ and $H$, we take $Q_c = 0$. By setting $x := \sigma_{i/4}(y)$ $(y = \sigma_{-i/4}(x))$, the expression of $Q$ in (2.14) $(Q_c = 0)$ equals that in (4.5).

Next, we substitute (4.5) into (4.2) to obtain

$$\begin{aligned} H &= D_{1/4}(y^*y) - 2\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*)) + D_{-1/4}(j(y^*y)) \\ &\quad -D_{1/4}(I_0(S(y^*y)) - D_{-1/4}(j(I_0(S(y^*y))). \end{aligned}$$

We use Lemma 3.1 and the fact that $I_0S = SI_0$ on $\mathcal{L}_{1/4}(\mathcal{H})$ to conclude that

$$\begin{aligned} H &= D_{1/4}T(I_0(y^*y)) - 2\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*)) + D_{-1/4}T(I_0(j(y^*y))) \\ &\quad -D_{1/4}S(I_0(y^*y)) + D_{-1/4}S(I_0(j(y^*y))) \\ &= 2I_0(y^*y) - 2T(I_0(\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*))) + 2I_0(j(y^*y)). \end{aligned} \tag{4.6}$$

Setting $y = \sigma_{-i/4}(x)$ and use (2.13). we get

$$\begin{aligned} TI_0(\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*)) &= I_0((D_{1/4}+D_{-1/4})x^*j(x^*)) \\ &= I_0(D_{-1/4}(x^*j(x^*))) + I_0(D_{1/4}(xj(x))) \\ &= I_0(\sigma_{-i/4}(x)^*j(\sigma_{-i/4}(x^*))) + I_0(\sigma_{-i/4}(x)j((\sigma_{-i/4}(x^*))^*). \end{aligned} \tag{4.7}$$

It follows from that Lemma 4.1 (a) that

$$2y^*y = \sigma_{-i/4}(x)^*\sigma_{-i/4}(x) + \sigma_{-i/4}(x^*)^*\sigma_{-i/4}(x^*). \tag{4.8}$$

We now substitute the equalities in (4.7) and (4.8) into (4.6) to conclude that

$$\begin{aligned} H \quad = \quad & I_0\left(\left[\sigma_{-i/4}(x) - j(\sigma_{-i/4}(x^*))\right]^*\left[\sigma_{-i/4}(x) - j(\sigma_{-i/4}(x^*))\right]\right) \qquad (4.9) \\ & + I_0\left(\left[\sigma_{-i/4}(x^*) - j(\sigma_{-i/4}(x))\right]^*\left[\sigma_{-i/4}(x^*) - j(\sigma_{-i/4}(x))\right]\right). \end{aligned}$$

By the definition of $I_0$ in (3.4), the expression of $H$ in (4.9) equals that in (2.7) with $f = f_0$. This proved Proposition 2.1 completely. □

*Proof of Theorem 2.2.* It is clear that Theorem 2.2 can be proved by the method used in the proof of Proposition 2.1. More precisely, if one replaces $y$ by $y_k$ and $x$ and $x_k$, respectively, in every expression containing $y$ and $x$ in the proof of Proposition 2.1 and sums over $k$ from 1 to $n$, and uses (2.16) and Lemma 4.1 (b) instead of (2.13) and Lemma 4.1 (a) respectively, then the proof of Theorem 2.2 follows from that of Proposition 2.1. □.

# 5 Discussion

We have seen that the conditions (2.16) and (2.17) are sufficient conditions such that the operator $H$ induced by the Lindblad type map $L$ given as (2.15) is self-adjoint and the seif-adjoint operator $H$ can be expressed as sum of Dirichlet operators given by (2.7) with $f = f_0$. We would like to give a brief discussion on the necessary and sufficient condition for self-adjointness of $H$ and also on the map $L$ on $\mathcal{M}$ associated to an Dirichlet operator (2.7) for a general admissible function $f$.

Let $L$ be given as (2.10) and let $H$ be the operator on $\mathcal{H}$ defined by (2.12). The necessary and sufficient condition for the self-adjointness of $H$ is given by (4.4) :

$$\begin{aligned} & iT(Q) - iT(j(Q)) \qquad (5.1) \\ & \quad = -S(y^*y) + S(j(y^*y)) + 2\sigma_{-i/4}(y^*)j(\sigma_{-i/4}(y^*)) - 2\sigma_{i/4}(y)j(\sigma_{i/4}(y)). \end{aligned}$$

From the above relation, one has to express $D_{1/4}(Q) - D_{-1/4}(j(Q))$ in terms of $y$ and $y^*$, and substitute it into (4.2). In general case, we are not able to estimate $Q$ from (5.1) directly. Thus we have assumed the property (2.13) to estimate $Q$ from (5.1).

If the state $\omega$ on $\mathcal{M}$ defined by $\omega(A) := \langle \xi_0, A\xi_0 \rangle$ is tracial, i.e., $\omega(AB) = \omega(BA)$, $\forall A, B \in \mathcal{M}$, then $\Delta = \mathbf{1}$ and the relation (5.1) becomes

$$iQ - ij(Q) = y^* j(y^*) - yj(y).$$

The above relation is equivalent to

$$i[Q, A] = y^* A y - y A y^*, \quad \forall A \in \mathcal{M}. \tag{5.2}$$

We substitute (5.2) into(2.10) and use (5.2) again with $A = \mathbf{1}$. Then the map $L$ given in (2.10) can be written as

$$L(A) = \frac{1}{2}\{y^* y A - 2y^* A y + A y^* y\} + \frac{1}{2}\{y y^* A - 2y A y^* + A y y^*\}$$

If one replace $y$ by $y_k$ in the above argument and sums over $k$, one can see that the Lindblad type generator (2.15) is symmetric if and only if $L$ can be written as

$$\begin{aligned} L(A) \quad &= \quad \frac{1}{2}\sum_{k=1}^{n}\{y_k^* y_k A - 2y_k^* A y_k + A y_k^* y_k\} \\ &\quad + \frac{1}{2}\sum_{k=1}^{n}\{y_k y_k^* A - 2y_k A y_k^* + A y_k y_k^*\}, \quad A \in \mathcal{M}. \end{aligned}$$

Notice that the condition (2.16) and (2.17) in Theorem 2.2 are satisfied automatically for the map $L$ in the above. Thus if $\xi_0$ defines a tracial state, the condition (2.16) and the condition (2.17) ($Q = 0$) are also necessary conditions for the map $L$ in (2.15) to be symmetric, or equivalently the operator $H$ induced by $L$ to be self-adjoint.

Next, consider a Dirichlet operator (2.7) for a given $x \in \mathcal{M}_{1/2}$ and an admissible function $f$ in the sense of Definition 2.1. Let $L : \mathcal{M} \to \mathcal{M}$ be the map given by

$$L(A) = L^{(1)}(A) + L^{(2)}(A), \tag{5.3}$$

where

$$\begin{aligned} L^{(1)}(A) \quad &= \quad \frac{1}{2}\int \Big\{\sigma_{t+i/4}(x^*)\sigma_{t-i/4}(x^*)A - 2\sigma_{t+i/4}(x^*)A\sigma_{t-i/4}(x) \\ &\qquad + A\sigma_{t+i/4}(x^*)\sigma_{t-i/4}(x)\Big\}\left(f(t-i/4) + f(t+i/4)\right)\, dt \\ &\quad + \frac{i}{2}[Q^{(1)}, A], \qquad\qquad (5.4) \\ Q^{(1)}(A) \quad &:= \quad i\int \left\{\sigma_t(x^*)\sigma_{t-i/2}(x) - \sigma_{t+i/2}(x^*)\sigma_t(x)\right\} f(t)\, dt, \end{aligned}$$

and $L^{(2)}(A)$ is defined as $L^{(1)}(A)$ replacing $x$ by $x^*$. Using the fact that for any $A \in \mathcal{L}_{1/4}(\mathcal{H})$

$$T\left(\int \sigma_t(A) f(t)\, dt\right) = \int \sigma_t(A)\left(f(t-i/4) + f(t+i/4)\right)\, dt,$$

and the method in the proof of Proposition 2.1, one can show that $L$ in (5.3) and $H$ in (2.7) is related by (2.12). We leave the detailed proof to the reader.

**Acknowledgements**: This work was supported by Korea Research Foundation (KRF-2003-005-C00010), Korean Ministry of Education.

# References

[1] L. Accardi, A. Frigerio, and J. T. Lewis, Quantum stochastic processes, *Publ. Res. Inst. Math. Sci.* **18**, 97-133 (1982).

[2] S. Albeverio and D. Goswami, A remark on the structure of symmetric Quantum Dynamical semigroups on von Neumann algebras, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, Vol. 3, No. 4, 571-579 (2002).

[3] S. Albeverio, R Høegh-Krohn and G. Olsen, Dynamical semigroups amd Markovian processes on C*-algebras, *J. Reine Angew. Math.* **319**, 25-37 (1980).

[4] L. Accardi, Topics in quantum probability, *Phys.Rep.* **77**, 169-192 (1981).

[5] H. Araki, Some properties of modular conjugation operator of von Neumann algebras ans noncommutative Radon-Nikodym theorem with chain rule, *Pacific J. Math.* **50** (2), 309-354 (1974).

[6] W. Arveson, Dynamical invariants for noncommutative flows, in *Operator algebras and quantum field theory* (Rome, 1996), Internat. Press, 476-514, (1997).

[7] C. Bahn and C. K. Ko, Construction of unbounded Dirichlet forms on standard forms of von Neumann Algebras, *J. Korean Math. Soc.* **39** (6) 931-951 (2002).

[8] C. Bahn, C. K. Ko amd Y. M. Park, Dirichlet forms and symmetric Markovian semigroups on CCR Algebras with quasi-free states, *J. Math. Phys.*, **44**, 723-753 (2003).

[9] C. Bahn, C. K. Ko amd Y. M. Park, Dirichlet forms and symmetric Markovian semigroups on $\mathbb{Z}_2$-graded von Neumann algebras, to be appeared in *Rev. Math. Phys.*

[10] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, Springer-Verlag, New York-Heidelberg-Berlin, vol I 1979, vol. II (1981).

[11] F. Cipriani, F. Fagnola and J. M. Lindsay, Spectral Analysis and Feller Properties for Quantum Ornstein-Uhlenbeck Semigroups, *Comm. Math. Phys.* **210**, 85-105 (2000).

[12] F. Cipriani and J. L. Sauvageot, Derivations as square roots of Diriclhlet forms, *J. Funct. Anal.* **201**, 78-120 (2003).

[13] F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, *J. Funct. Anal.* **147**, 259-300 (1997).

[14] E. B. Davies, *Quantum theory of open systems*, Academic Press, London-New York-San Francisco, (1976).

[15] D. Guido, T. Isola and S. Scarlatti, Non-commutative Dirichlet forms on semifinite von Neumann algebras, *J. Funct. Anal.* **135**, 50-75 (1996).

[16] S. Goldstein and J. M. Lindsay, KMS-symmetric Markov semigroups, *Math. Zeit.* **219**, 590-608 (1995).

[17] S. Goldstein and J. M. Lindsay, Markov semigroups KMS-symmetric for a weight, *Math. Ann.* **313**, 39-67 (1999).

[18] C. K. Ko and Y. M. Park, Construction of a Family of Quantum Ornstein-Uhlenbeck Semigroups, to be appeared in *J. Math. Phys.*

[19] G. Lindblad, On the generator on dynamical semigroups, *Comm. Math. Phys.* **48**, 119-130 (1976).

[20] A. W. Majewski and B. Zegarlinski, Quantum stochastic dynamics I: Spin systems on a lattice, *MPEJ* **1**, Paper 2 (1995).

[21] A. W. Majewski and B. Zegarlinski, Quantum stochastic dynamics II, *Rev. Math. Phys.* **8** (5), 689-713 (1996).

[22] T. Matsui, Markov semigroups on UHF algebras, *Rev. Mod. Phys.* **5**, 587-600 (1993).

[23] Y. M. Park, Construction of Dirichlet forms on standard forms of von Neumann algebras, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, Vol. 3, No. 1, 1-14 (2000).

[24] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser, Basel (1992).

[25] J. Quaegebeur, S. Stragler and A. Verbeure, Quantum detailed balance, *Ann. Inst. Henri Poincaré* **41** (1) 25-36 (1984).

[26] M.Reed and B.Simon, *Method of modern mathmatical physics I, II*, Academic press (1980).